Conformal Field Theory and Gravity

Solutions to Problem Set 10 Fall 2024

1. Coordinates and Isometries of AdS

(a) We will take X^0 and X^1 to be the timelike coordinates in the embedding spacetime. Global coordinates:

$$X^0 = \cosh(\rho)\cos(t), \tag{1}$$
$$X^1 = \cosh(\rho)\sin(t), \tag{2}$$
$$X^i = \sinh(\rho)\hat{n}^i, \quad (\hat{n}^i \text{ parameterizes } S^{D-2} \text{ with } \sum_{i=2}^{D+1}(\hat{n}^i)^2 = 1). \tag{3}$$

Global to Global 2:

$$r = \sinh(\rho) \tag{4}$$

Conformal coordinates:

$$X^0 = \frac{\cos(t)}{\cos(\theta)}, \tag{5}$$
$$X^1 = \frac{\sin(t)}{\cos(\theta)}, \tag{6}$$
$$X^i = \tan(\theta)\hat{n}^i. \tag{7}$$

Note that we require $\theta \in [0, \pi/2)$.
Poincaré Patch Coordinates:

$$X^0 = \frac{1}{2z}\left(1 + z^2 + \delta_{ij}x^i x^j - t^2\right), \tag{8}$$
$$X^1 = \frac{t}{z}, \tag{9}$$
$$X^i = \frac{x^i}{z}, \tag{10}$$
$$X^D = \frac{1}{2z}\left(1 - z^2 - \delta_{ij}x^i x^j + t^2\right). \tag{11}$$

Cosmological coordinates:

$$X^0 = \sin(T), \tag{12}$$
$$X^1 = \cos(T)\cosh(r), \tag{13}$$
$$X^i = \cos(T)\sinh(r)\hat{n}^i. \tag{14}$$

(b) Embedding space: $SO(2, D-1)$.
Global: time translations and rotations $SO(D-1)$.
Conformal: time translations and rotations $SO(D)$.
Poincaré: Poincaré group $ISO(1, D-2)$ and dilatations $z \to \lambda z, t \to \lambda t, x^i \to \lambda x^i$.
Cosmological: Hyperboloid group $SO(1, D-1)$.

(c) To find all the Killing vectors (KV) of AdS-Poincaré, our strategy is to locate the conformal Killing vectors (CKV) first, and then pick the KV from them.

Before analysing the details, we can first investigate the nature of the isometry group of AdS. We can see AdS$_D$ as a hyperboloid embedded in $\mathbb{R}^{2,D-1}$, so it inherits the isometry group $SO(2, D-1)$.

$$x^2 + y^2 + z^2 + \cdots - t^2 - u^2 = R_{\text{AdS}}^2$$

The $(D-1, 1)$-dimension Poincaré patch of AdS has metric:

$$ds^2 = \frac{dz^2 + \eta_{ij} dx^i dx^j}{z^2} = \frac{1}{z^2} ds_{\text{flat}}^2,$$

which is in the same conformal class as Minkowski, therefore, they have the same set of CKVs. The CKVs of Minkowski correspond to the $SO(2, D)$ algebra. For $A = 0, \cdots, D-1$, we have CKVs:

$$\begin{aligned}
P_A &= \frac{\partial}{\partial x^A}, \\
K_A &= 2x^A \left(x \cdot \frac{\partial}{\partial x} \right) - x^2 \frac{\partial}{\partial x^A}, \\
D &= x \cdot \frac{\partial}{\partial x}, \\
M_{AB} &= x^A \frac{\partial}{\partial x^B} - x^B \frac{\partial}{\partial x^A}.
\end{aligned}$$

Which of these are isometries of AdS-Poincaré? Or, instead, we can first pick the ones which aren't. P_z, M_{zi}, K_z are not (for $i = 0, \cdots, D-2, z = x^{D-1}$). Thus the KVs of AdS-Poincaré are:

$$\begin{aligned}
P_i &= \frac{\partial}{\partial x^i}, \\
K_i &= 2x^i \left(z \frac{\partial}{\partial z} + x^j \frac{\partial}{\partial x^j} \right) - \left(z^2 + \eta_{jk} x^j x^k \right) \frac{\partial}{\partial x^i}, \\
D &= z \frac{\partial}{\partial z} + x^i \frac{\partial}{\partial x^i}, \\
M_{ij} &= x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}.
\end{aligned}$$

At the boundary $z = 0$, these become:

$$\begin{aligned}
P_i &= \partial_i, \\
K_i &= 2x^i (x \cdot \partial) - x^2 \partial_i, \\
D &= x \cdot \partial, \\
M_{ij} &= x^i \partial_j - x^j \partial_i.
\end{aligned}$$

These are exactly the CKVs of $(D-1)$-dimensional Minkowski spacetime, which generates the group $SO(2, D-1)$.

2. Scalar field in AdS

(a) The separability of the wave function $\phi(t,\theta,\Omega)$ follows from the form of the Casimir $\Box$ in AdS. First, use that

$$\Box\phi = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu\phi) \tag{15}$$

and write $g_{\mu\nu}$ in the form

$$g_{\mu\nu} = \begin{pmatrix} -\frac{1}{\cos^2\theta} & 0 & 0 \\ 0 & \frac{1}{\cos^2\theta} & 0 \\ 0 & 0 & \frac{\sin^2\theta}{\cos^2\theta}h_{ij} \end{pmatrix} \tag{16}$$

where h_{ij} is the metric on the unit S^{d-1} sphere. Note that $\sqrt{|g|} = \frac{\sin^{d-1}\theta}{\cos^{d+1}\theta}\sqrt{|h|}$. Thus,

$$\begin{aligned}\Box\phi = &-\cos^2\theta\partial_t^2\phi + \frac{\cos^{d+1}\theta}{\sin^{d-1}\theta}\partial_r\left(\frac{\sin^{d-1}\theta}{\cos^{d+1}\theta}\cos^2\theta\partial_r\phi\right) \\ &+ \frac{\cos^2\theta}{\sin^2\theta}\underbrace{\frac{1}{\sqrt{|h|}}\partial_i(\sqrt{h}h^{ij}\partial_j\phi)}_{=\Delta_{S^{d-1}}\phi}\end{aligned} \tag{17}$$

where we recognized the sphere Laplacian $\Delta_{S^{d-1}}\phi$. This form is precisely of the form

$$\Box = \Box_t + \Box_r + \Box_{S^{d-1}} \tag{18}$$

The most general solution is given by a linear combination of solutions of the form $A(t)B(\theta)C(\Omega)$. The nice way to obtain solutions is to diagonalize each operator. We diagonalize ∂_t^2 in the standard way,

$$\partial_t^2\phi = -\omega^2\phi \tag{19}$$

where ω is the energy. The eigenfunctions of $\Delta_{S^{d-1}}$ are represented in spherical harmonics $Y_\ell(\Omega)$ (there are other quantum numbers as well. For example, in the case $d-1=2$, there would be $m=-\ell,...,\ell$) which obey the property

$$\Delta_{S^{d-1}}Y_\ell(\Omega) = -\ell(\ell+d-2)Y_\ell(\Omega) \tag{20}$$

You may be familiar with the case $d-1=2$, where $\Delta_{S^2} = -J^2$, $J^2 = \ell(\ell+1)$

(b) After having diagonalized $\partial_t^2 \to -\omega^2$ and $\Delta_{S^{d-1}} \to -\ell(\ell+d-2)$, $\Box\phi = m^2\phi$ reduces to

$$\cos^2\theta\, G''(\theta) + \frac{d-1}{\tan\theta}G'(\theta) + \left(\omega^2\cos^2\theta - \frac{\ell(\ell+d-2)}{\tan^2\theta}\right)G(\theta) = m^2G(\theta) \tag{21}$$

In the mathematica file provided, we find two independent solutions to this equation, namely

$$G_1(\theta) = \cos^{d-\Delta}\theta\ \sin^{\ell}\theta\ {}_2F_1\left(\frac{d+\ell-\Delta+\omega}{2}, \frac{d+\ell-\Delta-\omega}{2}, \frac{d}{2}-\Delta+1, \cos^2\theta\right) \tag{22}$$

$$G_2(\theta) = \cos^{\Delta}\theta\ \sin^{\ell}\theta\ {}_2F_1\left(\frac{\ell+\Delta+\omega}{2}, \frac{\ell+\Delta-\omega}{2}, -\frac{d}{2}+\Delta+1, \cos^2\theta\right) \tag{23}$$

These are not the two linearly independent solutions we were looking for, since we were looking for $\sin^2\theta$ as the last argument of ${}_2F_1$. To see what happens, we need to use hypergeometric function identities (c.f. wikipedia), including

$$\begin{aligned}{}_2F_1(a,b;c;z) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}{}_2F_1(a,b;a+b+1-c;1-z)\\ &\quad+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}{}_2F_1(c-a,c-b;1+c-a-b;1-z).\end{aligned} \tag{24}$$

Choosing $a=\frac{\ell+\Delta+\omega}{2}$, $b=\frac{\ell+\Delta-\omega}{2}$ and $c=-\frac{d}{2}+\Delta+1$, $z=\cos^2\theta$, $1-z=\sin^2\theta$, this implies that the geometric ${}_2F_1$ appearing in G_2 can be expressed as

$$\begin{aligned}{}_2F_1(a,b,c,\cos^2\theta) &= A\ {}_2F_1\left(\frac{\ell+\Delta+\omega}{2}, \frac{\ell+\Delta-\omega}{2}, \ell+\frac{d}{2}, \sin^2\theta\right)\\ &+B\sin^{-d-2l+2}\theta\ {}_2F_1\left(\frac{-d+\Delta-\ell+\omega+2}{2}, \frac{-d+\Delta-\ell-\omega+2}{2}, -\frac{d}{2}-\ell+2, \sin^2\theta\right)\end{aligned} \tag{25}$$

where A and B are come constants, independent of θ. Plugging this in G_2 this implies that

$$G_2(\theta) = AG_+(\theta) + BG_-(\theta) \tag{26}$$

where

$$G_+ = (\sin\theta)^{\ell}(\cos\theta)^{\Delta}{}_2F_1\left(\frac{\Delta+\ell+\omega}{2}, \frac{\Delta+\ell-\omega}{2}; \ell+\frac{d}{2}; \sin^2\theta\right) \tag{27}$$

$$G_- = (\sin\theta)^{(2-d-\ell)}(\cos\theta)^{\Delta}{}_2F_1\left(\frac{\Delta+2-d-\ell+\omega}{2}, \frac{\Delta+2-d-\ell-\omega}{2}; 2-\ell-\frac{d}{2}; \sin^2\theta\right) \tag{28}$$

The same analysis can be carried to G_1 with the use of the identity ${}_2F_1(a,b,c,z) = (1-z)^{c-a-b}{}_2F_1(c-a,c-b,c,z)$, as it should be, since G_+ and G_- are already independent solutions.

At $\theta\to 0$, ${}_2F_1(a,b,c,\sin^2\theta)=1$ and $\frac{d}{d\theta}\ {}_2F_1(a,b,c,\sin^2\theta)=0$ (where a, b, c are arbitrary), and similarily for $\cos^{\Delta}$. So the divergent behaviour of G_- only comes from $(\sin\theta)^{2-d-l}$,

$$\frac{d}{d\theta}G_-|_{\theta\to 0} \sim (2-d-\ell)\sin(\theta)^{(1-d-\ell)} \tag{29}$$

Since $d>1$ (the case of AdS_2 is special and we won't consider it), this diverges generically as $\theta\to 0$.

(c) The unit vector n^i pointing outwards the sphere has a non-zero component $n^\theta > 0$. To fix it, we impose the normalization

$$n^\mu g_{\mu\nu} n^\nu = 1 \implies n^\theta = \cos\theta \tag{30}$$

The induced metric on the fixed θ sphere boundary which we will denote by $\mathcal{B}$ is obtained by setting $d\theta = dt = 0$ in ds^2, thus

$$ds^2|_{\mathcal{B}} = \tan^2\theta \; h_{ij} d\alpha^i d\alpha^j \tag{31}$$

where α^i are $d-1$ angles on the unit sphere and h_{ij} is the metric on the unit sphere. This implies that

$$\sqrt{|g|_{\mathcal{B}}} = \sqrt{|h|}\tan^{d-1}\theta \tag{32}$$

Combining both, this gives

$$\int_{S^{d-1}} d^{d-1}\alpha \sqrt{|g|} n^i T_{it} = \int_{S^{d-1}} d^{d-1}\alpha \sqrt{|h|}\tan^{d-1}\theta \cos\theta \; T_{\theta t} \tag{33}$$

Note that $\tan^{d-1}\theta\cos\theta = \tan^{d-2}\theta \; \sin\theta$ and $\sin\theta \to 1$ as $\theta \to \pi/2$. Thus, the condition that it vanishes at the boundary is simply

$$\tan^{d-2}\theta \; T_{\theta t} \to 0 \qquad (\theta \to \pi/2) \tag{34}$$

Noting that $g_{\theta t} = 0$, $R_{\theta t} = 0$ (this follows from the fact that AdS_{d+1} is maximally symmetric, and thus $R_{\mu\nu} \propto g_{\mu\nu}$), the (θt) component of the stress tensor reads

$$T_{\theta t} = 2\partial_\theta\phi\partial_t\phi - \beta\nabla_\theta\nabla_t\phi^2 \tag{35}$$

Using $\partial_t\phi = -\omega^2\phi$ and $\Gamma^t_{\theta t} = \tan\theta$ is the only non-vanishing Christoffel with (θt) lower indices we obtain

$$T_{\theta t} = -\omega^2\partial_\theta\phi^2 + 2\beta\omega^2\partial_\theta\phi^2 - 2\beta\tan\theta\omega^2\phi^2 \tag{36}$$

Thus, the condition can be written as

$$(\tan\theta)^{d-2}[(1-2\beta)\partial_\theta + 2\beta\tan\theta]G_+^2 \to 0 \qquad (\theta \to \pi/2) \tag{37}$$

We now investigate the behaviour of G_+ when $\sin\theta \to 1$, $\cos\theta \to 0$. Let us define the arguments of the hypergeometric ${}_2F_1$ as

$$a \equiv \frac{\Delta + \ell + \omega}{2} \qquad b \equiv \frac{\Delta + \ell - \omega}{2} \qquad c \equiv \ell + \frac{d}{2} \tag{38}$$

One property of the hypergeometric is that, when $c < a + b$ (which is true in our case), the limit $z \to 1$ reads

$${}_2F_1(a,b,c,z) \sim (1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \tag{39}$$

Thus as $\sin\theta \to 1$,

$$\begin{aligned} G_+^2 &\sim (\sin\theta)^{2\ell}(\cos\theta)^{2\Delta}(\cos^2\theta)^{2(c-a-b)}\frac{\Gamma(c)^2\Gamma(a+b-c)^2}{\Gamma(a)^2\Gamma(b)^2} \\ &\sim (\cos\theta)^{2(d-\Delta)}\frac{\Gamma(c)^2\Gamma(a+b-c)^2}{\Gamma(a)^2\Gamma(b)^2} \end{aligned} \tag{40}$$

(we dropped $\sin\theta \sim 1$). For $\partial_\theta G_+^2$, one of the derivatives can hit $\cos\theta$ generating the term

$$\partial_\theta G_+^2 \sim (\cos\theta)^{2(d-\Delta)-1} \frac{\Gamma(c)^2\Gamma(a+b-c)^2}{\Gamma(a)^2\Gamma(b)^2} \tag{41}$$

Thus, the condition (36) (using $(\tan\theta) \sim (\cos\theta)^{-1}$) reduces to

$$(\cos\theta)^{d-2\Delta+1} \frac{\Gamma(c)^2\Gamma(a+b-c)^2}{\Gamma(a)^2\Gamma(b)^2} \to 0 \tag{42}$$

Note that $d - 2\Delta + 1 \leq -d + 1$, so $(\cos\theta)^{d-2\Delta+1} \to \infty$. We thus need

$$\frac{\Gamma(c)^2\Gamma(a+b-c)^2}{\Gamma(a)^2\Gamma(b)^2} = 0 \tag{43}$$

This is true whenever a or b is a pole of the gamma function (the gamma function has no zeros), namely

$$a = -n \quad \text{or} \quad b = -n \qquad n = 0, 1, ... \tag{44}$$

This implies

$$\Delta + \ell \pm \omega = -2n \tag{45}$$

meaning

$$\pm\omega = \Delta + \ell + 2n \tag{46}$$

(d) For ω to be real, we require Δ to be real. This means

$$d^2 + 4m^2 > 0 \implies m^2 \geq -\frac{d^2}{4}\,. \tag{47}$$

Plugging back the AdS radius R, this would give

$$m^2R^2 \geq -\frac{d^2}{4}\,. \tag{48}$$

This is known as the Breitenlohner-Freedman bound.

3. Lowest scalar energy state in AdS

(a) *(optional part)* The CFT generators are expressed in terms of the $SO(d,2)$ generators as

$$D = -J_{0,d+1}, \quad M_{\mu\nu} = J_{\mu\nu}, \tag{49}$$

$$P_\mu = J_{\mu 0} + iJ_{\mu,d+1}, \quad K_\mu = J_{\mu 0} - iJ_{\mu,d+1}, \tag{50}$$

with

$$J_{MN} = -i\left(X_M \frac{\partial}{\partial X^N} - X_N \frac{\partial}{\partial X^M}\right). \tag{51}$$

where X^A are the embedding coordinates which in global coordinates read

$$X^0 = R\cos t\cosh\rho, \tag{52}$$
$$X^\mu = R\Omega^\mu\sinh\rho, \tag{53}$$
$$X^{d+1} = -R\sin t\cosh\rho. \tag{54}$$

It's easiest to express the derivative $\partial/\partial t$ in terms of the $\partial/\partial X^A$ (and likewise for $\partial/\partial\rho$ and ∇_μ) instead of working the other way around. For instance, we have

$$\frac{\partial}{\partial t} = \frac{\partial X^A}{\partial t}\frac{\partial}{\partial X^A} = -R\sinh\rho\frac{\partial}{\partial X^0} - R\cosh t\cosh\rho\frac{\partial}{\partial X^{d+1}}. \tag{55}$$

But this is precisely

$$\frac{\partial}{\partial t} = X^{d+1}\frac{\partial}{\partial X^0} - X^0\frac{\partial}{\partial X^{d+1}} = -X_{d+1}\frac{\partial}{\partial X^0} + X_0\frac{\partial}{\partial X^{d+1}} = iJ_{0,d+1}, \tag{56}$$

since $X_0 = -X^0$ and $X_{d+1} = -X^{d+1}$. Hence $-J_{0,d+1} = i\partial_t$, as claimed.
Next, if we extend the Ω^μ to cover all of $\mathbb{R}^d$, we have

$$\frac{\partial}{\partial\Omega^\mu} = \frac{X^A}{\partial\Omega^\mu}\frac{\partial}{\partial X^A} = R\sinh\rho\frac{\partial}{\partial X^\mu}, \quad \mu = 1,\ldots,d, \tag{57}$$

so more properly speaking, we have the $\mathbb{S}^{d-1}$-covariant derivatives

$$\nabla_\mu = R\sinh\rho\left[\frac{\partial}{\partial X^\mu} - \Omega_\mu\Omega^\nu\frac{\partial}{\partial X^\nu}\right] \tag{58}$$

which indeed obey $\Omega^\mu\nabla_\mu = 0$. Consequently

$$X_\mu\frac{\partial}{\partial X_\nu} = R\sinh\rho\,\Omega_\mu\frac{\partial}{\partial X^\nu} = \Omega_\mu\nabla_\nu + \Omega_\mu\Omega_\nu\Omega^\rho\frac{\partial}{\partial X^\rho}, \tag{59}$$

so (since the second term is symmetric in $\mu\leftrightarrow\nu$)

$$J_{\mu\nu} = -i\left(\Omega_\mu\nabla_\nu - \Omega_\nu\nabla_\mu\right) = M_{\mu\nu}. \tag{60}$$

Finally, we have

$$\frac{\partial}{\partial\rho} = \frac{\partial X^A}{\partial\rho}\frac{\partial}{\partial X^A} = R\left[\cos t\sinh\rho\frac{\partial}{\partial X^0} + \Omega^\mu\cosh\rho\frac{\partial}{\partial X^\mu} - \sin t\sinh\rho\frac{\partial}{\partial X^{d+1}}\right]. \tag{61}$$

We are aiming for an expression of the form J_{AB}, expressed purely in terms of the X_A. As a first step, we can introduce the operators

$$\mathcal{D}_\mu = \Omega_\mu\frac{\partial}{\partial\rho} + \frac{1}{\tanh\rho}\nabla_\mu \tag{62}$$

which are somewhat simpler: they can be recast as

$$\mathcal{D}_\mu = R\left[\cos t\sinh\rho\,\Omega_\mu\frac{\partial}{\partial X^0} + \cosh\rho\frac{\partial}{\partial X^\mu} - \Omega_\mu\sin t\sinh\rho\frac{\partial}{\partial X^{d+1}}\right], \tag{63}$$

$$= X_\mu \left(\cos t \frac{\partial}{\partial X^0} - \sin t \frac{\partial}{\partial X^{d+1}} \right) + R \cosh \rho \frac{\partial}{\partial X^\mu}. \tag{64}$$

Moreover

$$\Omega^\mu \tanh \rho \frac{\partial}{\partial t} = -X_\mu \left(\sin t \frac{\partial}{\partial X^0} + \cos t \frac{\partial}{\partial X^{d+1}} \right). \tag{65}$$

Using the identity $R \cosh \rho = -e^{\pm it}(X_0 \pm iX_{d+1})$, we can therefore write

$$\mathcal{D}_\mu - i\Omega_\mu \tanh \rho \frac{\partial}{\partial t} = e^{it} \left[X_\mu \left(\frac{\partial}{\partial X^0} + i \frac{\partial}{\partial X^{d+1}} \right) - (X_0 + iX_{d+1}) \frac{\partial}{\partial X^\mu} \right], \tag{66}$$

that is to say

$$-ie^{-it} \left(\mathcal{D}_\mu - i\Omega_\mu \tanh \rho \frac{\partial}{\partial t} \right) = J_{\mu 0} + iJ_{\mu, d+1}. \tag{67}$$

Likewise

$$\mathcal{D}_\mu + i\Omega_\mu \tanh \rho \frac{\partial}{\partial t} = e^{-it} \left[X_\mu \left(\frac{\partial}{\partial X^0} - i \frac{\partial}{\partial X^{d+1}} \right) - (X_0 - iX_{d+1}) \frac{\partial}{\partial X^\mu} \right], \tag{68}$$

that is to say

$$-ie^{it} \left(\mathcal{D}_\mu + i\Omega_\mu \tanh \rho \frac{\partial}{\partial t} \right) = J_{\mu 0} - iJ_{\mu, d+1}. \tag{69}$$

This agrees precisely with the formulas from the exercise.

(b) The requirement $D\phi = \Delta\phi$ gives

$$i \frac{\partial}{\partial t} \phi = \Delta \phi \implies \phi(r, t, \Omega) = e^{-i\Delta t} \phi_0(r, \Omega) \tag{70}$$

The requirement $K_\mu \phi = 0$ implies that $\nabla_\mu \phi = 0$ (independent of the S^{d-1} angular coordinates), and

$$(-\partial_\rho - i \tanh \rho \partial_t) \phi_0 = 0 \implies \partial_\rho \phi_0 = -\Delta \tanh \rho \ \phi_0 \tag{71}$$

This is solved by

$$\phi_0 \propto e^{-\Delta \log \cosh \rho} \implies \phi \propto \left(\frac{e^{-it}}{\cosh \rho} \right)^\Delta \tag{72}$$

(c) The change of variables between the two coordinates system is $\tanh \rho = \sin \theta \implies \frac{1}{\cosh \rho} = \cos \theta$. So we obtained that a scalar operator ϕ which is a **primary state of the CFT algebra with dimension** Δ is expressed as

$$\phi_{\text{CFT,primary}} \propto e^{-i\Delta t} (\cos \theta)^\Delta \tag{73}$$

whereas in the last exercise we solved the **massive scalar fields in** AdS and obtained that $\phi = e^{-i\omega t} G_+(\theta) Y_\ell(\Omega)$. Choosing $\ell = 0$, $\omega = \Delta_m$ (**lowest energy state**) and $Y_{\ell=0} \propto 1$ the solutions reads

$$\phi_{AdS,\text{lowest energy state}} \propto e^{-i\Delta_m t} (\cos \theta)^{\Delta_m} {}_2F_1(1, 0, \frac{d}{2}, \sin^2 \theta) = e^{-i\Delta_m t} (\cos \theta)^{\Delta_m} \tag{74}$$

where Δ_m in this exercise was defined as $\Delta_m \equiv \frac{d}{2}(1 + \sqrt{1 + 4m^2})$. We see that the two are the same, upon identifying the CFT dimension Δ with the quantity Δ_m.